Tutorial 11

The following exercise is quoted from Yuval Peres's book *Game theory, Alive*, page 135.

Exercise 1. *There are n players. Player 1 has two left gloves, while each of the other $n-1$ players has one right glove. For a coalition S, let $\nu(S)$ be the number of pairs of gloves that can be formed from the gloves owned by the members of S.*

(i) For $n = 3$, find the Shapley values.

(ii) Find the Shapley values for a general n.

Solution. (i) Let the player set be $\{1, 2, 3\}$. Then we have $\nu(\{1, 2\}) = \nu(\{1, 3\}) = 1$, $\nu(\{1, 2, 3\}) = 2$ and $\nu(S) = 0$ for any other subset S of $\{1, 2, 3\}$. Hence

$$\phi_1 = \frac{1}{3!}[(3-2)!(2-1)! \times 1 \times 2 + (3-3)!(3-1)! \times 2] = 1$$

and

$$\phi_2 = \frac{1}{3!}[(3-2)!(2-1)! \times (1-0) + (3-3)!(3-1)! \times (2-1)] = \frac{1}{2}.$$

It is clear that Play 2 and Player 3 are symmetric. Hence $\phi_3 = \phi_2 = \frac{1}{2}$.

(ii) For general n, let the player set be $\{1, \cdots, n\}$. It is easy to see that

$$\nu(S) = \begin{cases} 1 & \text{if } S = \{1, k\}, k = 2, \cdots, n, \\ 2 & \text{if } 1 \in S \text{ and } \#S \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned}
\phi_1 &= \frac{1}{n!}\left[\sum_{1\in S,\#S=2}(n-2)!(2-1)!\times(1-0)+\sum_{1\in S,\#S\geq 3}(n-|S|)!(|S|-1)!\times(2-0)\right]\\
&= \frac{1}{n!}\cdot(n-1)!+\frac{2}{n!}\sum_{1\in S,\#S\geq 3}(n-|S|)!(|S|-1)!\\
&= \frac{1}{n}+\frac{2}{n!}\sum_{k=3}^{n}\sum_{1\in S,\#S=k}(n-k)!(k-1)!\\
&= \frac{1}{n}+\frac{2}{n!}\sum_{k=3}^{n}\binom{n-1}{k-1}(n-k)!(k-1)!\\
&= \frac{1}{n}+\frac{2}{n!}\cdot(n-2)(n-1)! = 2-\frac{3}{n}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\phi_2 &= \frac{1}{n!}\sum_{1\in S,2\in S}(n-|S|)!(|S|-1)!(\nu(S)-\nu(S\setminus\{2\}))\\
&= \frac{1}{n!}\cdot(n-2)!(2-1)!\times(1-0)+\frac{1}{n!}\sum_{1\in S,2\in S,\#S=3}(n-3)!(3-1)!\times(2-1)\\
&= \frac{1}{n(n-1)}+\frac{1}{n!}\cdot 2(n-3)!(n-2)\\
&= \frac{3}{n(n-1)}.
\end{aligned}$$

By symmetricity, we have

$$\phi_2=\phi_3=\cdots=\phi_n=\frac{3}{n(n-1)}.$$

Exercise 2 (The glove market game)**.** *Let $\mathcal{A}$ be the set of players. Assume that there are two types P and Q of players. That is $\mathcal{A}=P\cup Q$ and $P\cap Q=\emptyset$. For any coalition $S\subseteq\mathcal{A}$, define*

$$\nu(S)=\min\{|S\cap P|,|S\cap Q|\},$$

where $|\cdot|$ *denotes the cardinality of a set. The game* $(\mathcal{A}, \nu)$ *is called the glove market game.*

(i) If $|P| = |Q| = 2$*, find the core* $C(\nu)$*.*

(ii) If $|P| = 2$ *and* $|Q| = 3$*, find* $C(\nu)$*.*

(iii) Find $C(\nu)$ *for general* P *and* Q*.*

Solution. (i) For convenience, let us assume that $\mathcal{A} = \{1, 2, 3, 4\}$, $P = \{1, 2\}$ and $Q = \{3, 4\}$. Then $\nu(\mathcal{A}) = 2$. Moreove, by the characterization of the core, it is easy to see that a point $(x_1, x_2, x_3, x_4) \in C(\nu)$ if and only if

$$\begin{cases} x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, & \text{(1a)} \\ x_1 + x_3 \geq 1, x_1 + x_4 \geq 1, x_2 + x_3 \geq 1, x_2 + x_4 \geq 1, & \text{(1b)} \\ x_1 + x_2 + x_3 + x_4 = 2. & \text{(1c)} \end{cases}$$

From (1b) we get $2(x_1 + x_2 + x_3 + x_4) \geq 4$, this combining with (1c) implies that all the inequalities in (1b) are indeed equalities, i.e.

$$x_1 + x_3 = x_1 + x_4 = x_2 + x_3 = x_2 + x_4 = 1.$$

Equivalently,

$$x_1 = x_2, x_3 = x_4, x_1 + x_3 = 1.$$

Hence the core is given by

$$C(\nu) = \{(x, x, 1 - x, 1 - x) : 0 \leq x \leq 1\}.$$

(ii) Assume that $\mathcal{A} = \{1, 2, 3, 4, 5\}$, $P = \{1, 2\}$ and $Q = \{3, 4, 5\}$. Clearly

$\nu(\mathcal{A}) = 2$. It is easy to see that $(x_1, x_2, x_3, x_4, x_5) \in C(\nu)$ if and only if

$$\begin{cases} x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, & \text{(2a)} \\ \begin{cases} x_1 + x_3 \geq 1, x_1 + x_4 \geq 1, x_1 + x_5 \geq 1, \\ x_2 + x_3 \geq 1, x_2 + x_4 \geq 1, x_2 + x_5 \geq 1, \end{cases} & \text{(2b)} \\ x_1 + x_2 + x_3 + x_4 + x_5 = 2. & \text{(2c)} \end{cases}$$

Take summation over all the inequalities in (2b), we get

$$3x_1 + 3x_2 + 2x_3 + 2x_4 + 2x_5 \geq 6.$$

Equivalently,

$$x_1 + x_2 + 2(x_1 + x_2 + x_3 + x_4 + x_5) \geq 6.$$

This inequality combining with (2c) implies that $x_1 + x_2 \geq 2$. Now apply (2c) again, we get $x_3 = x_4 = x_5 = 0$. By (2b), we have $x_1 = x_2 = 1$. Hence in this case the core is the singleton given by

$$C(\nu) = \{(1, 1, 0, 0, 0)\}.$$

(iii) For general P and Q, we assume that $|P| = n$ and $|Q| = m$. We consider two cases separately as follows.

Case 1. $|P| = |Q| = n$. In this case, assume $\mathcal{A} = \{1, \cdots, n, n+1, \cdots, 2n\}$, $P = \{1, \cdots, n\}$ and $Q = \{n+1, \cdots, 2n\}$. Clearly $\nu(\mathcal{A}) = n$. We have

$(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}) \in C(\nu)$ if and only if

$$\begin{cases} x_i \geq 0, 1 \leq i \leq 2n, & \text{(3a)} \\ \begin{cases} x_1 + x_{n+1} \geq 1, \cdots, x_1 + x_{2n} \geq 1, \\ x_2 + x_{n+1} \geq 1, \cdots, x_2 + x_{2n} \geq 1, \\ \quad \vdots \qquad\qquad \cdots \qquad \vdots \\ x_n + x_{n+1} \geq 1, \cdots, x_n + x_{2n} \geq 1, \end{cases} & \text{(3b)} \\ x_1 + \cdots + x_{2n} = n. & \text{(3c)} \end{cases}$$

Take summation over the inequalities in (3b), we get $n(x_1 + \cdots + x_{2n}) \geq n^2$. Hence by (3c) we deduce that all the inequalities in (3b) are indeed equalities. Consequently,

$$x_1 = \cdots = x_n, x_{n+1} = \cdots = x_{2n} \text{ and } x_1 + x_{n+1} = 1.$$

Hence the core is

$$C(\nu) = \left\{ (\overbrace{x, \cdots, x}^{n \text{ terms}}, \overbrace{1 - x, \cdots, 1 - x}^{n \text{ terms}}) : 0 \leq x \leq 1 \right\}.$$

Case 2. $|P| = n$, $|Q| = m$ and $n \neq m$. Without loss of generality, we assume that $n < m$. Let $\mathcal{A} = \{1, \cdots, n, n+1, \cdots, n+m\}$, $P = \{1, \cdots, n\}$ and $Q = \{n+1, \cdots, n+m\}$. Note that $\nu(\mathcal{A}) = n$. By a similar argument as in Case 1, we can find that the core is given by the singleton

$$C(\nu) = \left\{ (\overbrace{1, \cdots, 1}^{n \text{ terms}}, \overbrace{0, \cdots, 0}^{m \text{ terms}}) \right\}.$$